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Periodic Solutions of Some Nonlinear Degenerate Parabolic Equations

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INTRODUCTION

In this paper we are concerned with the existence and estimates of periodic solutions for some nonlinear degenerate parabolic equations. We first consider the problem

$$\begin{aligned} &u_t - \Delta\beta(u) + B(x, t, u) = f(x \cdot t) \qquad \text{in} \quad \Omega \times R \\ &u|_{\partial\Omega} = 0 \qquad \text{and} \qquad u(x, t) = u(x, t + \omega) \end{aligned} \tag{1}$$

where Ω is a bounded domain in R^n with smooth (e.g., C^3-class) boundary $\partial\Omega$, $\beta(u)$ is a function of the form

$$\beta(u) = \int_0^u g(\eta^2)\, d\eta$$

and B, f are functions with period ω in t.

The case $\beta(u) = u$ or $\beta'(u) > 0$ for problem (1) has been considered by many authors (Prodi [14], Fife [7], Smulev [16], Kusano [9], Biroli [5], Vaghi [17], Bange [3] and Nakao [11], etc.) In particular, if we assume that g, B and f are appropriately smooth, $\beta'(u) = g(u^2) \geqslant \mu(L) > 0$ for $|u| \leqslant L$ and

$$B(x, t, u)\, u \geqslant b_0 |u|^2 - b_1$$

for some positive constants b_0 and b_1 it is well known that (1) has a classical solution $u(x, t)$ (cf. Smulev [16]).

The purpose of this paper is to show the existence of a solution and to estimate it precisely under a weaker assumption which admits $\beta'(0) = 0$. For $B(x, t, u)$ we require only

$$B(x, t, u)\, u \geqslant -b_0 |u|.$$

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Our method is as follows. First we derive an L^∞ estimate for approximate solutions u_ε by the use of Moser's technique (cf. Alikakos [1]). On the basis of this estimate we next show L^2-estimates for some derivatives of $\beta(u)$. Then standard compactness argument yields a desired solution.

Our approach is applicable to another type of nonlinear degenerate equation. In the last section we shall discuss briefly the typical problem

$$u_t - \sum_{i=1}^{n} \frac{\partial}{\partial x_i}\left(\left|\frac{\partial}{\partial x_i} u\right|^m \frac{\partial}{\partial x_i} u\right) = f(x,t) \qquad (m>0)$$
$$u|_{\partial\Omega} = 0 \qquad \text{and} \qquad u(t) = u(t+\omega). \tag{2}$$

The existence of the (generalized) solution of (2) belonging to $W^{1,2}(\omega; L^2) \cap L^\infty(\omega; W_0^{1,m+1})$ is well known (Seidman [15]), while here we show that the solution belongs to $L^\infty(\Omega \times R)$ if $f \in L^\infty(\Omega \times R)$. Also we derive a precise L^∞ estimate as well as a $W_0^{1,m+1}$ estimate.

Throughout the paper we denote by $L^p(\omega; X)$ the set of ω-periodic X-valued measurable functions f with

$$\left(\int_0^\omega \|f(t)\|_X^p \, dt\right)^{1/p} < \infty.$$

Similar notation will be employed freely.

1. Assumptions and Results

Here we state precise assumptions on β, B and f and the result.

A.1. $\beta(u)$ is of the form

$$\beta(u) = \int_0^u g(\eta^2)\, d\eta$$

with a function $g \in C([0,\infty)) \cap C^{1+\alpha}((0,\infty))$, $0<\alpha<1$, such that

$$g(\eta^2) \geqslant C_0 |\eta|^m$$

for some $m \geqslant 0$ and a positive constant C_0. Moreover it holds that

$$\varlimsup_{u\to 0} \frac{|\beta(u)|^2}{\beta'(u)} < \infty.$$

A.2. $B(x, t, u)$ is a Hölder continuous function defined on $\bar{\Omega} \times R \times R$ with period ω in t and satisfies

$$B(x, t, u)\, u \geqslant -b_0 |u|$$

for some $b_0 > 0$.

A.3. $f \in L^\infty(\omega; L^\infty(\Omega))$. We set $M = \text{ess.sup}_{x,t} |f(x, t)|$.

In what follows we denote L^p norm on Ω by $\|\cdot\|_p$. Our result reads as follows.

THEOREM 1. *Under the assumptions* A.1–A.3 *the problem* (1) *has a solution u such that*

$$u \in L^\infty(\omega; L^\infty), \quad \beta(u) \in L^\infty(\omega; \mathring{H}_1) \quad \textit{and} \quad \frac{\partial}{\partial t} \int_0^u \sqrt{g(\eta^2)}\, d\eta \in L^2(\omega; L^2)$$

where the equation is satisfied in the sense that

$$\int_0^\omega \int_\Omega \{-u\phi_t + \nabla\beta(u)\,\nabla\phi + B(x, t, u)\,\phi - f\phi\}\, dxdt = 0 \tag{3}$$

$\forall\, \phi \in C^1(\omega; \mathring{H}_1)$. *In particular we have the estimate*

$$\|u(t)\|_\infty \leqslant C \left(\frac{M + b_0}{\delta}\right)^{\delta + 1/(m+1)}$$

for $0 < \forall\, \delta \ll 1$ *and for some* $C > 0$ *independent of* M *and* δ.

COROLLARY 1. *In addition to* A.1–A.3 *assume*

$$\beta'(u) > 0 \qquad \forall\, u.$$

Then the solution in Theorem 1 *is classical.*

THEOREM 2. *Let* $\beta(u) = |u|^m u$, $f \in L^2(\omega; L^2)$ *and* $B(x, t, u) = 0$. *Then* (1) *has a unique solution* $u(t)$ *such that*

$$\frac{\partial}{\partial t}(|u|^{m/2}\, u) \in L^2(\omega; L^2) \qquad \textit{and} \qquad |u|^m u \in L^\infty(\omega; \mathring{H}_1)$$

and the following estimates hold.

$$\|\nabla\beta(u(t))\|_2^2 \leqslant C\{\bar{M}^{4(m+1)/(m+1)} + \bar{M}^2\}$$

and

$$\int_0^\omega \left\| \frac{\partial}{\partial t} (|u|^{m/2} u) \right\|_2^2 ds \leqslant C(\bar{M}^{(3m+4)/(m+2)} + \bar{M}^{(3m+2)/(m+1)})$$

where we set

$$\bar{M} = \left(\int_0^\omega \|f(t)\|_{2(m+1)}^2 \, dt \right)^{1/2}.$$

Concerning the second problem (2) we obtain.

THEOREM 3. *Let $f \in L^2(\omega; L^2)$ and $u(t)$ be the unique solution of (2) with $u \in C(\omega; W_0^{1,m+2}) \cap W^{1,2}(\omega; L^2)$. Then*

$$\|u(t)\|_{W_0^{1,m+2}} \leqslant CM_0^{1/(m+1)} \qquad \text{for some } C > 0$$

and

$$\left(\int_0^\omega \|u_t\|_2^2 \, dt \right)^{1/2} \leqslant M_0$$

where we set

$$M_0 = \left(\int_0^\omega \|f(t)\|_2^2 \, dt \right)^{1/2}.$$

THEOREM 4. *Let $f \in L^\infty(\omega; L^\infty(\Omega))$. Then the solution $u(t)$ belongs also to $L^\infty(\omega; L^\infty(\Omega))$, and $\forall\, \delta$ with $0 < \delta \leqslant 1$ we have*

$$\|u(t)\|_\infty \leqslant C \left(\frac{M}{\delta} \right)^{\delta + 1/(m+1)}.$$

2. L^∞ ESTIMATE OF APPROXIMATE SOLUTIONS

We assume $f(x, t)$ is Hölder continuous. This assumption is removed easily at the final step. To construct approximate smooth solutions of (1) we consider the modified problem for $\varepsilon > 0$.

$$\begin{gathered} u_t - \Delta \beta_\varepsilon(u) + \varepsilon u + B(x, t, u) = f \\ u|_{\partial\Omega} = 0 \qquad \text{and} \qquad u(x, t) = u(x, t + \omega) \end{gathered} \tag{4}$$

where we set

$$\beta_\varepsilon(u) = \int_0^u g(\eta^2 + \varepsilon) \, d\eta.$$

Since $\beta_\varepsilon'(u)=g(u^2+\varepsilon)\geqslant C_0\,\varepsilon^{m/2}$ (by A.1) the problem (4) admits a classical solution u_ε. Applying a classical technique we have easily

$$\|u_\varepsilon(t)\|_\infty \leqslant (b_0+M)/\varepsilon.$$

For our purpose, however, we must prove that $\|u_\varepsilon\|_\infty$ is bounded by a constant independent of ε. To do so we employ the so-called Moser's technique as in [1].

LEMMA 2.1. *For all $\varepsilon>0$ there exists a constant independent of ε and δ such that*

$$\|u_\varepsilon(t)\| \leqslant C\{\delta^{-1}(M+b_0)\}^{\delta+1/(m+1)}.$$

Proof. We write u for u_ε. Multiplying (4) by $|u|^p u$ $(p\geqslant 0)$ and integrating in x we have

$$\frac{1}{p+2}\frac{d}{dt}\|u(t)\|_{p+2}^{p+2}+(p+1)\int_\Omega g(u^2+\varepsilon)\,|\nabla u|^2\,|u|^p\,dx$$
$$+\int_\Omega B(x,t,u)\,|u|^p\,u dx=\int_\Omega |u|^p\,u f dx \tag{5}$$

and, using the assumptions on g and B,

$$\frac{d}{dt}\|u(t)\|_{p+2}^{p+2}+C\,\|\nabla(|u|^{(p+m)/2}\,u)\|_2^2$$
$$\leqslant (p+2)\,\|u\|_{p+2}^{p+1}(\|f(t)\|_{p+2}+Cb_0) \tag{6}$$

where C denotes various positive constants independent of p and ε. We set

$$p_k=2^k+m-2,\qquad \alpha_k=2(p_k+2)/(p_k+m+2)\qquad \text{and}$$
$$w_k=|u|^{(p_k+m)/2}u,\qquad (k=1,2,3,\ldots).$$

Then, from (6),

$$\frac{d}{dt}\|w_k\|_{\alpha_k}^k+C\,\|\nabla w_k\|_2^2\leqslant C(p_k+2)\|w_k\|_{\alpha_k}^{\alpha_k(p_k+1)/(p_k+2)}(M+b_0). \tag{7}$$

Here we appeal to the Gagliardo–Nirenberg inequality

$$\|w_k\|_{\alpha_k}\leqslant C\,\|\nabla w_k\|_2^{\theta_k}\,\|w_k\|_1^{1-\theta_k} \tag{8}$$

with

$$\theta_k=\left(1-\frac{1}{\alpha_k}\right)\left(\frac{1}{N}-\frac{1}{2}+1\right)^{-1}=\frac{p_k-m+2}{(p_k+2)}\cdot\frac{N}{N+2}.$$

From (7), (8) and the fact that $\|w_k\|_1 = \|w_{k-1}\|_{\alpha_{k-1}}^{\alpha_{k-1}}$ we arrive at the differential inequality

$$\frac{d}{dt}\|w_k(t)\|_{\alpha_k}^{\alpha_k} \leqslant \{-C\|w_k(t)\|_{\alpha_k}^{2/\theta_k-\alpha_k(p_k+1)/(p_k+2)}\, x_{k-1}^{2\alpha_{k-1}(1-1/\theta_k)} \\ + C(p_k+2)(M+b_0)\}\,\|w_{\alpha_k}(t)\|_{\alpha_k}^{\alpha_k(p_k+1)/(p_k+2)} \tag{9}$$

where we set $x_{k-1} = \sup_t \|w_{k-1}(t)\|_{\alpha_{k-1}}$.

Taking the periodicity of $\|w_k(t)\|_{\alpha_k}$ into account we can obtain from (9)

$$\|w_k(t)\|_{\alpha_k} \leqslant [C(p_k+2)(M+b_0)\, x_{k-1}^{2\alpha_{k-1}\theta_k/(1-\theta_k)}]^{\eta_k} \tag{10}$$

where

$$\eta_k = \left[\frac{2}{\theta_k} - \frac{\alpha_k(p_k+1)}{p_k+2}\right]^{-1}.$$

Noting $\eta_k \to N/4$ (>0) and $p_k 2^{-k} \to 1$ as $k \to \infty$, we have from (10)

$$x_k \leqslant C\, 2^{k\eta_k}(M+b_0)^{\eta_k} x_{k-1}^{\nu_k} \tag{11}$$

with $\nu_k = 2\alpha_{k-1}(1-\theta_k)\,\eta_k/\theta_k$.

From (11),

$$\begin{aligned} \log x_k \leqslant{} & \left(1 + \sum_{i=k_0+1}^{k} \nu_i \cdots \nu_k\right) \log C \\ & + \left(\sum_{i=k_0+1}^{k-1} \eta_i \nu_{i+1} \cdots \nu_k\right) \log(M+b_0) \\ & + \left(\sum_{i=k+1}^{k-1} i\eta_i \nu_{i+1} \cdots \nu_k\right) \log 2 \\ & + \nu_k \cdots \nu_{k_0+1} \log x_{k_0}. \end{aligned} \tag{12}$$

Now, it is easy to see that there exists $k_0 > 0$ such that for $k \geqslant k_0$,

$$2\left(1 - \frac{2(m+1)}{p_k}\right) \leqslant \nu_k \leqslant 2\left(1 - \frac{1}{3p_k}\right) \tag{13}$$

and

$$\frac{N}{4}\left(1 - \frac{C}{p_k}\right) \leqslant \eta_k \leqslant \frac{N}{4}. \tag{14}$$

The most delicate term in the right-hand side of (12) is the third one, which is treated as follows.

$$\sum_{i=k_0+1}^{k-1} i\eta_i \nu_{i+1} \cdots \nu_k \leqslant \frac{N}{4} \sum_{i=k_0+1}^{k-1} i 2^{k-i-1} = 2^{k-2} N \sum_{i=k_0+1}^{k-1} \frac{i}{2^{i+1}}$$
$$\leqslant C k_0 2^{k-k_0}.$$

Similarly we can handle other terms of (12) to obtain

$$\log x_k \leqslant 2^{k-k_0} \left(\bar{x}_{k_0} + \frac{N}{4} C(k_0) \log(M+b_0) + C k_0 \log 2 + \log C \right) \quad (15)$$

where $\bar{x}_k = \max(x_k, 1)$ and $C(k_0)$ is a constant such that $\lim_{k_0 \to \infty} C(k_0) = 1$ (more precisely it may be different corresponding to the cases $x_{k_0}, M + b_0 < 1$ and $x_{k_0}, M + b_0 \geqslant 1$).

Since

$$x_k = \sup_t \| u(t) \|_{p_k+2}^{2^{k-1}+m}$$

we obtain from (15)

$$\sup_t \| u(t) \|_\infty \leqslant \limsup_{k \to \infty} \sup \| u(t) \|_{p_k+2}$$
$$\leqslant C(M+b_0)^{C(k_0) 2^{-1-k_0} N} \bar{x}_{k_0}^{2^{1-k_0}}. \quad (16)$$

On the other hand, by (7) and Sobolev's Lemma, we have

$$\frac{d}{dt} \| w_{k_0} \|_{\alpha_{k_0}}^{\alpha_{k_0}} + C \| w_{k_0} \|_{\alpha_{k_0}}^2 \leqslant C(p_{k_0}+2) \| w_{k_0} \|_{\alpha_{k_0}}^{\alpha_{k_0}(p_{k_0}+1)/(p_{k_0}+2)} \times (M+b_0)$$

which implies

$$\| w_{k_0}(t) \|_{\alpha_{k_0}} \leqslant [C(p_{k_0}+2)(M+b_0)]^{(p_{k_0}+2)/[2(p_{k_0}+2) - \alpha_{k_0}(p_{k_0}+1)]}$$
$$= [C(2^{k_0}+m)(M+b_0)]^{(2^{k_0-1}+m)/(m+1)}. \quad (17)$$

Since we may take k_0 arbitrarily large the inequalities (16) and (17) imply

$$\| u(t) \|_\infty \leqslant C \{ \delta^{-1} (M+b_0) \}^{\delta + 1/(m+1)}$$

for $0 < \forall \delta \ll 1$. Q.E.D.

3. Further Estimation for u and the Proof of Theorem 1

To see the convergence of $\{u_\varepsilon\}$ as $\varepsilon \to 0$ we prepare the following

Lemma 3.1.

$$\|\nabla \beta_\varepsilon(u_\varepsilon(t))\|_2^2 \leqslant C\{(M+b_0)^2 + L_0 + L_1\} + O(\varepsilon)$$

and

$$\int_0^\omega \int_\Omega \beta_\varepsilon'(u_\varepsilon)\,|u_{\varepsilon_t}|^2\,dxds \leqslant L_1 + O(\varepsilon)$$

where we set

$$L_0 = \sup_{\substack{0 \leqslant |u| \leqslant K_\delta \\ x,t}} \beta'(u(x,t))(|B(x,t,u)| + |f(x,t)|)^2$$

and

$$L_1 = \sup_{\substack{0 < |u| \leqslant K_\delta \\ x,t}} |\beta(u)|^2\,\beta'(u)^{-1} \qquad \text{with} \quad K_\delta = C(\delta^{-1}M)^{\delta + 1/(m+1)}.$$

Proof. We write u for u_ε again. Multiplying, (4) by $(\partial/\partial t)\,\beta_\varepsilon(u)$,

$$\frac{d}{dt}\left\{\frac{1}{2}\|\nabla\beta_\varepsilon(u)\|_2^2 + \varepsilon\iint_0^u \eta\beta_\varepsilon'(\eta)\,d\eta\,dx\right\} + \int \beta_\varepsilon'(u)\,|u_t|^2\,dx$$

$$= \int \beta_\varepsilon'(u)\,u_t F\,dx \leqslant \frac{1}{2}\int \beta_\varepsilon'(u)\,|u_t|^2\,dx + \frac{1}{2}\int \beta_\varepsilon'(u)\,|F|^2\,dx$$

and hence

$$\frac{d}{dt}\left(\|\nabla\beta_\varepsilon(u)\|_2^2 + 2\varepsilon\iint_0^u \eta\beta_\varepsilon'(\eta)\,d\eta dx\right) + \int \beta_\varepsilon'(u)\,|u_t|^2\,dx \leqslant CL_0, \qquad (18)$$

where we set $F(x,t) = -B(x,t,u(x,t)) + f(x,t)$.

On the other hand, multiplying the equation by $\beta_\varepsilon(u)$,

$$\|\nabla\beta_\varepsilon(u)\|_2^2 + \varepsilon\int \beta_\varepsilon(u)\,dx$$

$$\leqslant \int |\beta_\varepsilon u_t|\,dx + (M+b_0)\int \beta_\varepsilon(u)\,dx$$

$$\leqslant (\beta_\varepsilon'(u)\,|u_t|^2\,dx)^{1/2}\left(\int \beta_\varepsilon^2(\beta_\varepsilon')^{-1}\,dx\right)^{1/2} + C(M+b_0)^2$$

$$+ \frac{1}{2}\|\nabla\beta_\varepsilon(u)\|_2^2$$

and

$$\|\nabla\beta_\varepsilon(u)\|_2^2 + 2\varepsilon\int\beta_\varepsilon(u)\,dx \leqslant \int\beta'_\varepsilon(u)\,|u_t|^2\,dx + C(M+b_0)^2 + CL_1 + O(\varepsilon). \tag{19}$$

Adding the inequalities (18) and (19) we have

$$\frac{d}{dt}\left(\|\nabla\beta_\varepsilon(u(t))\|_2^2 + 2\varepsilon\iint_0^{u(t)}\eta\beta'_\varepsilon(\eta)\,d\eta\,dx\right) \leqslant -\left(\|\nabla\beta_\varepsilon(u(t))\|_2^2 + 2\varepsilon\iint_0^{u}\eta\beta'_\varepsilon(\eta)\,d\eta\,dx\right) + C\{L_0 + L_1 + (M+b_0)^2\} + O(\varepsilon)$$

which implies

$$\|\nabla\beta_\varepsilon(u(t))\|_2^2 \leqslant C\{L_0 + L_1 + (M+b_0)^2\} + O(\varepsilon). \tag{20}$$

We have also from (18)

$$\int_0^\omega\int\beta'_\varepsilon(u(t))\,|u_t|^2\,dxdt \leqslant CL_0. \quad \text{Q.E.D.} \tag{21}$$

Proof of Theorem 1. After Lemmas 2.1 and 3.1 are established the proof of Theorem 1 is easy (cf. Aronson *et al.* [18], Nakao [13], etc.). Indeed, setting $U_\varepsilon = \beta_\varepsilon(u_\varepsilon)$ it is easily seen that $\{U_\varepsilon\}$ and $\{(\partial/\partial t)\,U_\varepsilon\}$ are bounded in $L^\infty(\omega; \mathring{H}_1)$ and $L^2(\omega; L^2)$, respectively. Thus we can extract a subsequence U_ε and a function $U(x,t)$ such that

$$U_\varepsilon \to U \qquad \text{strongly in } L^2(\omega; L^2) \text{ and a.e. in } \Omega \times R$$

and

$$U_\varepsilon \to U \qquad \text{weakly* in } L^\infty(\omega; L^2).$$

Setting $u(x,t) = \beta^{-1}(U(x,t))$ we see by Lemma 3.1

$$\frac{\partial}{\partial t}\int_0^{u_\varepsilon}\sqrt{\beta'_\varepsilon(\eta)}\,d\eta \to \frac{\partial}{\partial t}\int_0^{u}\sqrt{\beta'(\eta)}\,d\eta \qquad \text{weakly in } L^2(\omega; L^2).$$

Thus $u(x,t)$ is a desired solution of (1). The estimates follow from the Lemmas immediately. Finally, we note that the Hölder continuity of f is removed by approximating f by smooth function $f_\varepsilon(x,t)$ with $|f_\varepsilon(x,t)| \leqslant M + O(\varepsilon)$.

The proof of Corollary 1 is clear, because if $\beta'(u)$ is nondegenerate the estimates of higher-order derivatives of u are derived by a standard method on the basis of the estimate for $\|u(t)\|_\infty$ (cf. Smulev [16]).

4. Proof of Theorem 2

The existence of a solution follows immediately from the estimates below. (Cf. [13]) We give a rather formal calculation, which is easily justified through appropriate approximate solutions.

Similarly as in the proof of Lemma 3.1 multiplication by $(\partial/\partial t)\beta(u)$ yields

$$\int_0^\omega \int_\Omega \beta'(u)\,|u_t|^2\,dxdt \leqslant \int_0^\omega \beta'(u)\,|f|^2\,dxds$$
$$\leqslant C\int_0^\omega \|\nabla\beta(u)\|_2^{m/(m+1)}\,\|f\|_{2(m+1)}^2\,dt. \tag{22}$$

Next, multiplying by $\beta(u)$,

$$\int_0^\omega \|\nabla\beta(u(t))\|_2^2\,dt = \int_0^\omega f(t)\,\beta(u(t))\,dxdt$$
$$\leqslant \frac{1}{2}\int_0^\omega \|\nabla\beta(u)\|_2^2 + \frac{1}{2}\int_0^\omega \|f\|_2^2\,dt$$

and

$$\int_0^\omega \|\nabla\beta(u(t))\|_2^2 \leqslant \int_0^\omega \|f(t)\|_2^2\,dt. \tag{23}$$

The inequality (23) implies that there exists $t^* \in [0,\omega]$ such that

$$\|\nabla\beta(u(t^*))\|_2^2 \leqslant \frac{1}{\omega}\int_0^\omega \|f(t)\|_2^2\,dt.$$

Thus, multiplying the equation by $(\partial/\partial t)\beta(u)$ and integrating over $\Omega\times[t^*, t+\omega]$

$$\|\nabla\beta(u(t))\|_2^2 \leqslant \|\nabla\beta(u(t^*))\|_2^2 + \int_{t^*}^{t+\omega}\int \beta'(u)(|u_t|^2 + |f|)\,dxds$$
$$\leqslant C\left\{\int_0^\omega (\|\nabla\beta(u(t))\|_2^{m/(m+1)}\,\|f(t)\|_{2(m+1)}^2 + \|f\|_2^2)\,dt\right\}$$

$\forall\, t \in [0, \omega]$, and

$$\|\nabla\beta(u(t))\|_2^2 \leqslant C\left\{\sup_t \|\nabla\beta(u(t))\|_2^{m/(m+1)} \int_0^\omega \|f(t)\|_{2(m+1)}^2 + \int_0^\omega \|f(t)\|_2^2\, dt\right\}$$

which implies

$$\sup \|\nabla\beta(u(t))\|_2^2 \leqslant C\{\bar{M}^{2(m+1)/(m+2)} + \bar{M}^2\}. \tag{24}$$

Hence we have also

$$\int_0^\omega \int_\Omega \beta'(u)\,|u_t|^2\, dxdt \leqslant C\{\bar{M}^{(3m+4)(m+2)} + \bar{M}^{(3m+2)/(m+1)}\}.$$

The solution satisfies

$$\frac{d}{dt}u(t) - \Delta\beta(u(t)) = f(t) \qquad \textit{in } H^{-1}$$

and the operator $-\Delta\beta(\cdot)$ is strictly monotone in H^{-1} (cf. Lions [10], Brezis [6], Nakao [13], etc.). Uniqueness of periodic solution follows from this fact.

5. Another Equation

Here we give the proofs of Theorems 3 and 4. Let $f \in L^2(\omega; L^2)$. Then, it is well known (see Seidman [15], Barbu [4], etc.) that there exists a unique solution of the problem (2) such that

$$u \in W^{1,2}(\omega; L^2) \cap C(\omega; W_0^{1,m+1}).$$

Thus our task is to derive the estimates for this solution.

Proof of Theorem 3. The proof is parallel to that of Theorem 2. Multiplying the equation by $(\partial/\partial t)\, u$ and u we have

$$\int_0^\omega \|u_t\|_2^2\, dt \leqslant \int_0^\omega \|f(t)\|_2^2\, dt \tag{25}$$

and

$$\int_0^\omega \|u(t)\|_{W_0^{1,m+2}}^{m+2}\, dt \leqslant C \int_0^\omega \|f(t)\|_2^{(m+2)/(m+1)}\, dt, \tag{26}$$

respectively. From (25) and (26) we can obtain as in (24) (cf. [12])

$$\sup_t \|u(t)\|_{W_0^{1,m+2}} \leqslant \|f\|^{1/(m+2)}_{L^2(\omega;L^2)}. \qquad \text{Q.E.D.}$$

Proof of Theorem 4. We give rather formal proof, which is justified by the theory of nonlinear semi-groups (cf. Barbu [4], Attouch and Damlamian [2], Herrero and Vazquez [8], etc.).

Multiplying the equation under consideration by $|u|^p u$ we have as in (7)

$$\begin{aligned}\frac{d}{dt}\|w_k\|^{\alpha_k}_{\alpha_k} &+ C(p_k+m+2)^{-m}\|\nabla w_k\|^{m+2}_{m+2}\\ &\leqslant C(p_k+2)\|w_k\|^{\alpha_k(p_k+1)/(p_k+2)}_{\alpha_k} M \end{aligned}\tag{27}$$

where we set at this time

$$w_k = |u|^{p_k/(m+2)}u,$$

$$p_k = (m+2)\,p_{k-1} + m + 2 = [(m+2)^{k+1} - (m+2)]/(m+1)$$

and

$$\alpha_k = (m+2)(p_k+2)/(p_k+m+2) \qquad (k=1,2,3,...).$$

Using the Gagliardo–Nirenberg inequality

$$\|w_k\|_{\alpha_k} \leqslant C\|\nabla w_k\|^{\theta_k}_{m+2}\|w_k\|^{1-\theta_k}_1$$

with

$$\theta_k = \frac{(m+1)\,p_k+m+2}{p_k+2}\cdot\frac{N}{(m+1)\,N+2}$$

and the relation $\|w_k\|_1 = \|w_{k-1}\|^{\alpha_{k-1}}_{\alpha_{k-1}}$, we have (see (9))

$$\begin{aligned}\frac{d}{dt}\|w_k\|^{\alpha_k}_{\alpha_k} \leqslant \{&-C(p_k+m+2)^{-m}\|w_k\|^{(m+2)/\theta_k - \alpha_k(p_k+1)/(p_k+2)}_{\alpha_k}\\ &\times x_{k-1}^{(m+2)\alpha_{k-1}(1-1/\theta_k)}\\ &+ C(p_k+2\,M\}\,\|w_k\|^{\alpha_k(p_k+1)/(p_k+2)}_{\alpha_k}\end{aligned}\tag{28}$$

with $x_k = \sup_t \|w_k(t)\|_{\alpha_k}$. From (28),

$$x_k \leqslant C(p_k+m+2)^{(m+1)\eta_k} M^{\eta_k} x_{k-1}^{\nu_k} \tag{29}$$

where

$$\eta_k = [(m+2)/\theta_k - \alpha_k(p_k+1)/(p_k+2)]^{-1}$$

and

$$\nu_k = [(m+2)\,\alpha_{k-1}(1-\theta_k)\,\eta_k/\theta_k].$$

Noting $\eta_k = (m+1)N/(m+2)^2 + O(1/p_k)$ and $\nu_k = (m+2) + O(1/p_k)$ as $k \to \infty$, we obtain from (29) (see (16))

$$\|u(t)\|_\infty \leqslant CM^{C(k_0)(m+2)^{-k_0+1}} x_{k_0}^{C(k_0)(m+1)(m+2)^{-k_0}} \tag{30}$$

for large k_0, where $C(k_0)$ is a constant satisfying $\lim_{k_0 \to} C(k_0) = 1$. Moreover, (27) together with Sobolev's lemma implies

$$x_{k_0} \leqslant C[(p_{k_0}+2)\,M]^{\{(m+2)^{k_0}+m\}/(m+1)^2}. \tag{31}$$

From (30) and (31) we can conclude

$$\|u(t)\|_\infty \leqslant C\left(\frac{M}{\delta}\right)^{\delta+1/(m+1)}$$

for $0 < \forall\delta \ll 1$. Q.E.D.

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